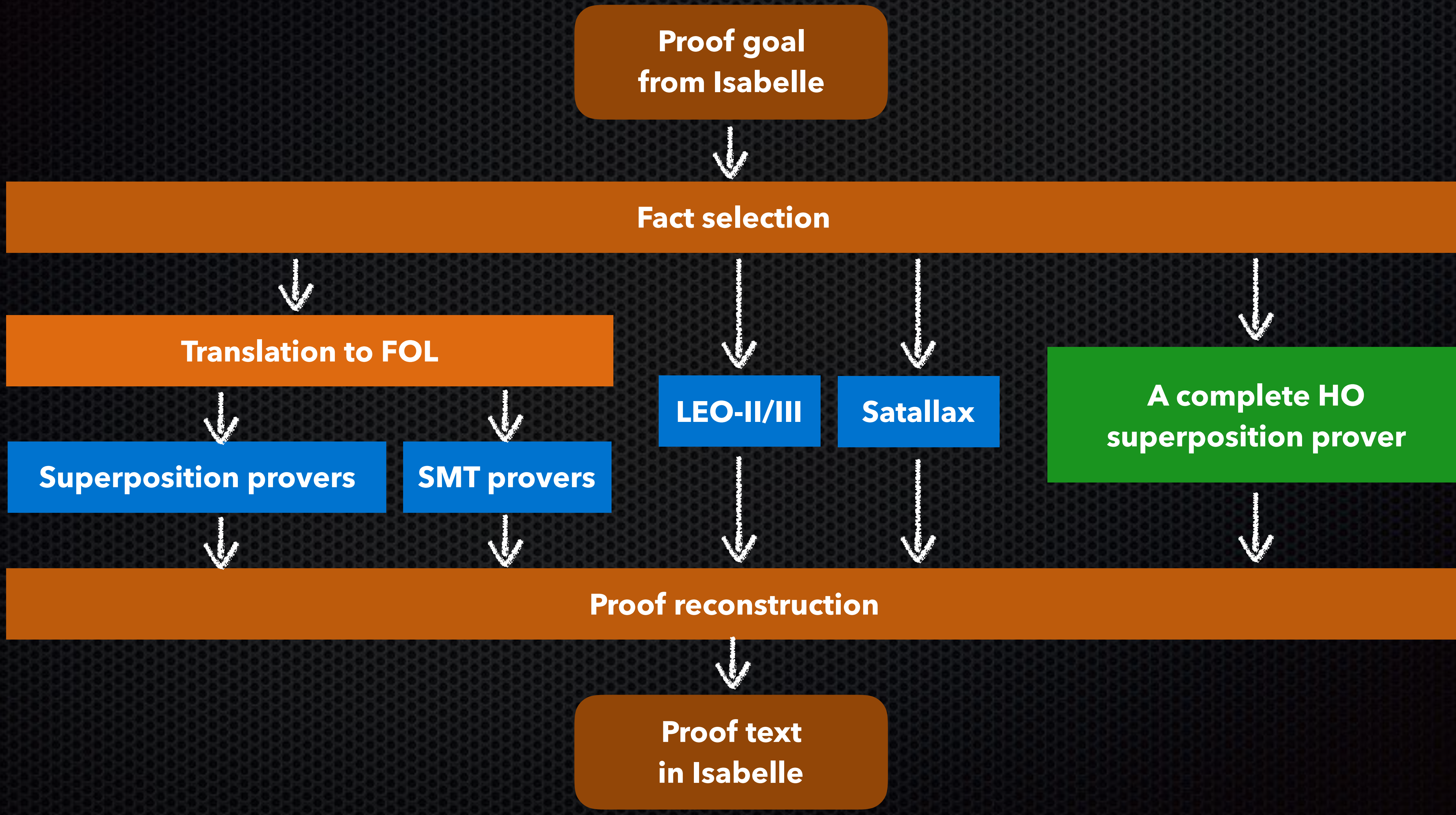
Superposition with Lambdas

Alexander Bentkamp
Jasmin Blanchette
Sophie Tourret
Petar Vukmirović
Uwe Waldmann

Motivation: Sledgehammer



Milestones towards HOL

FOL

partial application
& applied variables

λ-free HOL /
applicative FOL

λ-expressions /
comprehension
axioms

Boolean-free
HOL

boolean formulas
nested in terms

HOL

Challenges

#1 Higher-order unification

#2 Superposition below applied variables

#3 No ground-total simplification order

#1 Higher-Order Unification

- **Undecidability & no most general unifier**
  - Our approach: dovetailing
- **Flex-flex pairs**
  - Huet's preunification algorithm requires constrained clauses
  - Our approach: Jensen & Pietrzykowski's algorithm
  - Future work: More efficient unification algorithms (complete or incomplete)

#2 Applied Variables

$f\ a = c$ $\qquad$ $h\ (X\ a)\ (X\ b) \neq h\ (g\ c)\ (g\ (f\ b))$

Superposition
"half below" a variable?

Unsatisfiable because:

$$X \mapsto \lambda u.\ g\ (f\ u)$$

yields

$$h\ (g\ (f\ a))\ (g\ (f\ b)) \neq h\ (g\ c)\ (g\ (f\ b))$$

$= c$

#2 Applied Variables

f a = c

add artificial context

Y (f a) = Y c

h (X a) (X b) ≠ h (g c) (g (f b))

superpose

Unifier of Y (f a) and X a :

$Y \mapsto \lambda u.\ Z\ a\ u\ u$

$X \mapsto \lambda v.\ Z\ v\ (f\ v)\ (f\ a)$

h (Z a c c) (Z b (f b) (f a)) ≠ h (g c) (g (f b))

This is a new inference rule: FluidSup

#3 No Ground-Total Simplification Order

$(\lambda x.\, x) > (\lambda x.\, b)$

Then, by compatibility with contexts:

$a = (\lambda x.\, x)\ a > (\lambda x.\, b)\ a = b$

or

$(\lambda x.\, x) < (\lambda x.\, b)$

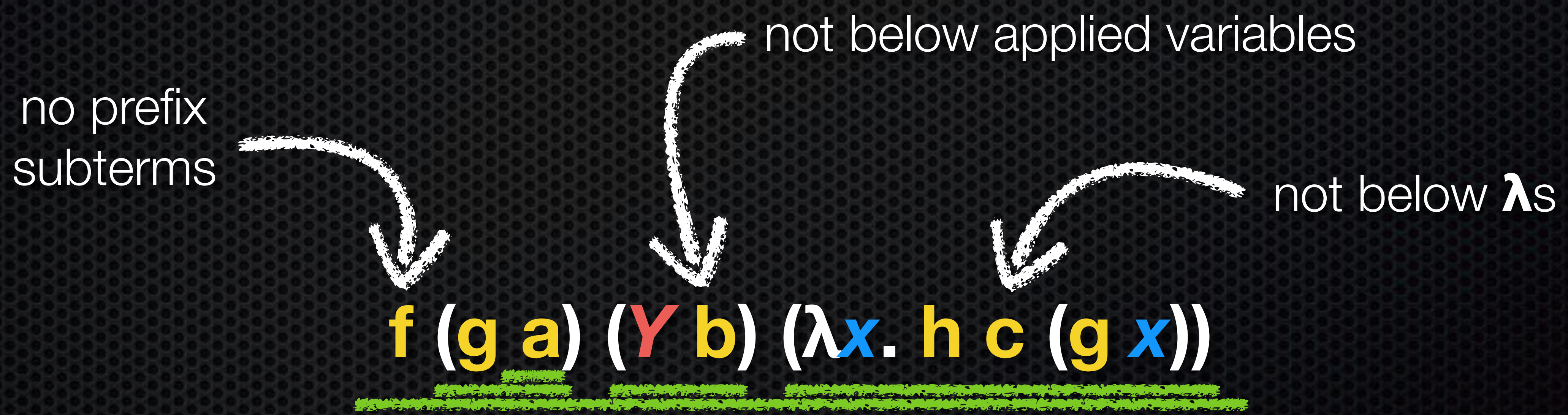
Then, by compatibility with contexts:

$c = (\lambda x.\, x)\ c < (\lambda x.\, b)\ c = b$

?

#3 No Ground-Total Simplification Order

Our solution:
Compatibility only with green contexts



Superposition only at green subterms
ArgCong, FluidSup, and the extensionality axiom access other subterms

Our Calculus

$$\frac{D \lor t = t' \qquad C \lor [\neg]\ s[u] = s'}{(D \lor C \lor [\neg]\ s[t'] = s')\sigma}\ \text{Sup} \quad \sigma \in CSU(t,u)$$

$$\frac{C \lor s' = t' \lor s = t}{(C \lor t \neq t' \lor s = t')\sigma}\ \text{EqFact} \quad \sigma \in CSU(s,s')$$

$$\frac{D \lor t = t' \qquad C \lor [\neg]\ s[u] = s'}{(D \lor C \lor [\neg]\ s[Z\ t'] = s')\sigma}\ \text{FluidSup} \quad \sigma \in CSU(Z\ t,u)$$

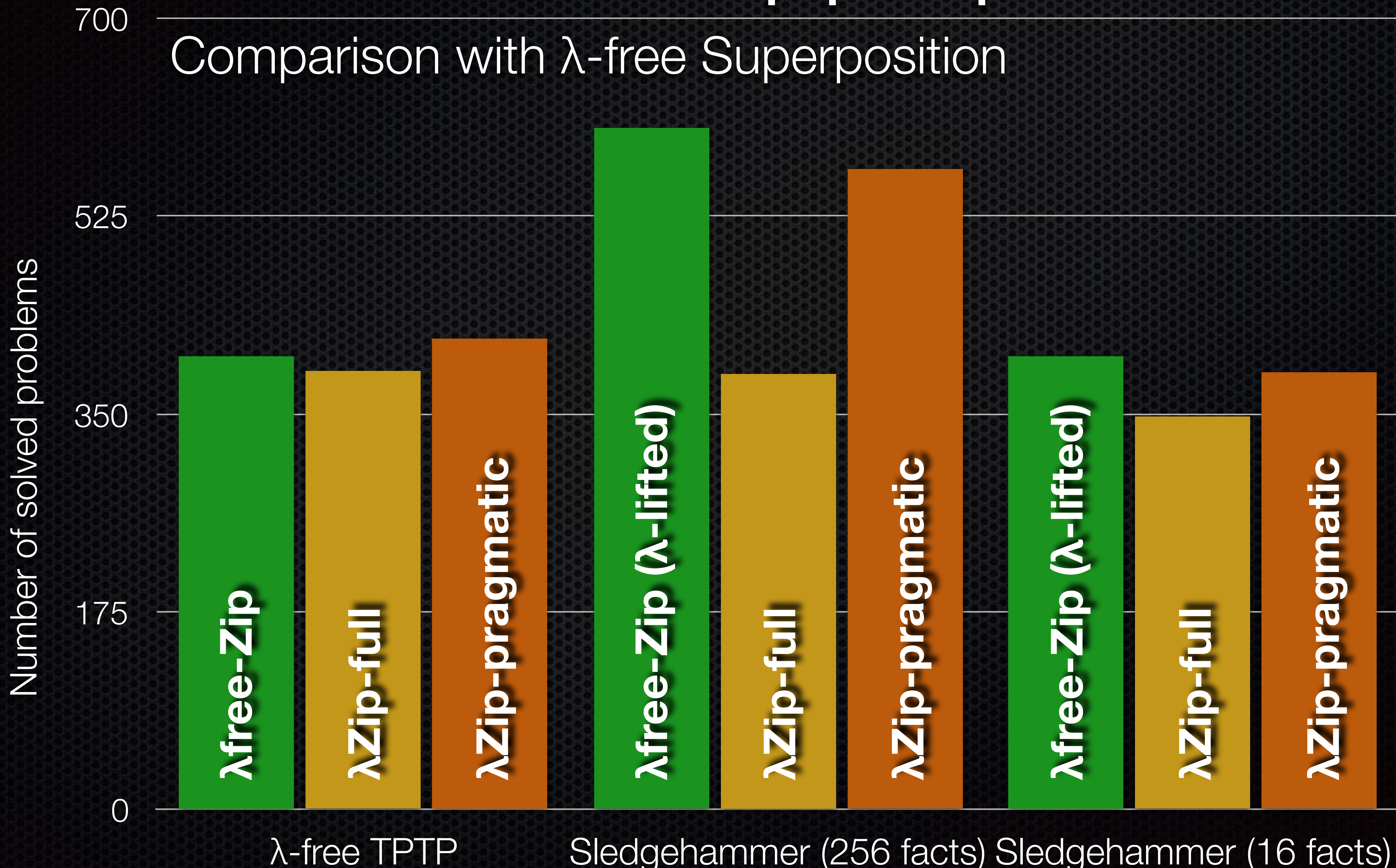
$$\frac{C \lor s \neq t}{C\sigma}\ \text{EqRes} \quad \sigma \in CSU(s,t)$$

$$\frac{C \lor s = t}{C \lor (s\sigma)\ \bar{X} = (t\sigma)\ \bar{X}}\ \text{ArgCong}$$

$$\frac{}{X\ (\text{diff}\ X\ Y) \neq Y\ (\text{diff}\ X\ Y) \lor X = Y}\ \text{Ext}$$

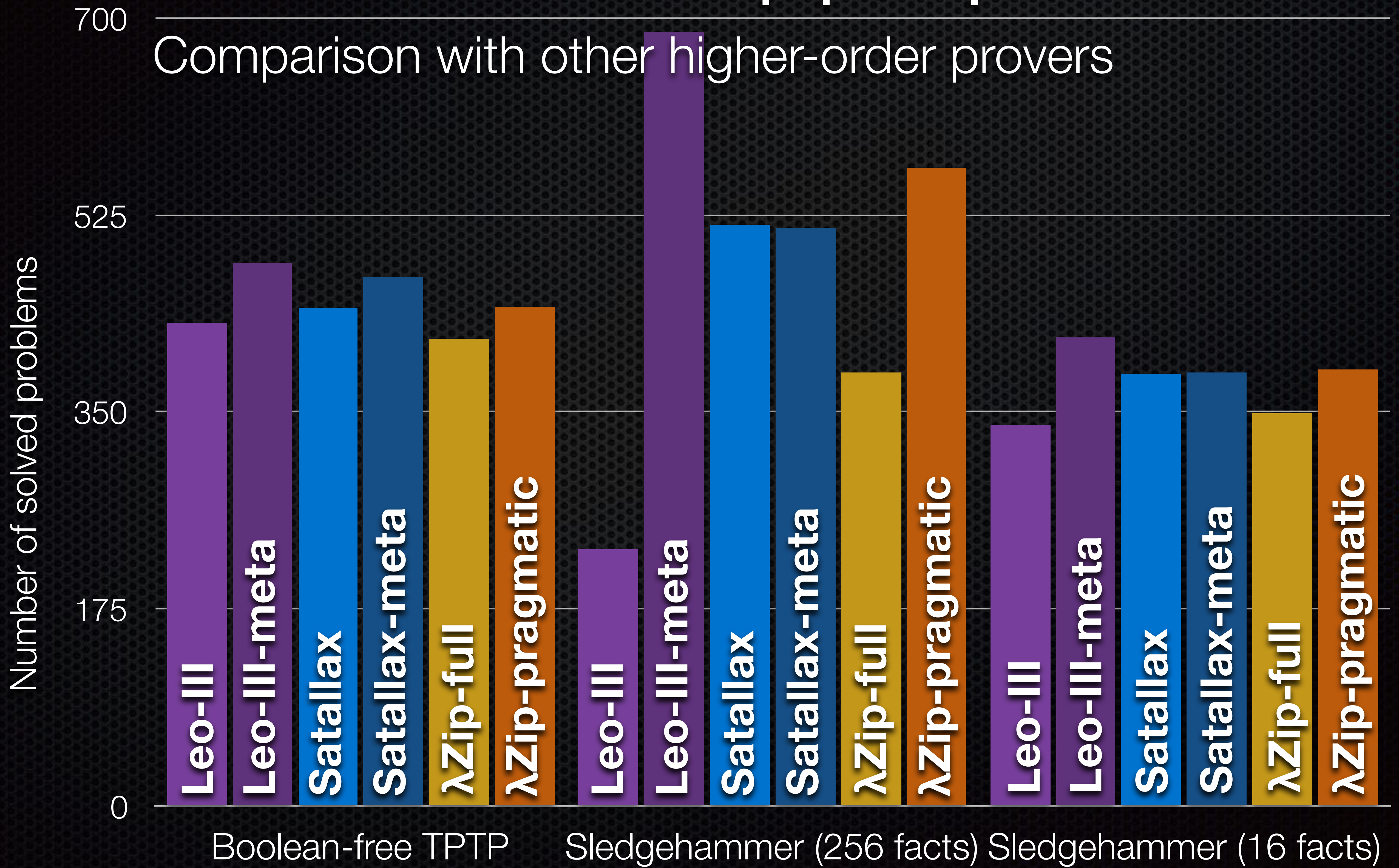
All clauses are kept in β-normal η-short form.

Evaluation in Zipperposition



Evaluation in Zipperposition

Comparison with other higher-order provers



Summary

- Complete superposition calculus for Boolean-free HOL
- Promising experimental results for an incomplete variant of this calculus
- Many remaining challenges:
  - First-class Boolean type
  - More efficient unification
  - More efficient treatment of extensionality
  - More efficient alternatives to FluidSup
  - Implementation in E