The Embedding Path Order for λ-free higher-order terms

Alexander Bentkamp

Recursive Path Order (FOL)

Let $t >_{\mathsf{fo}} s$ if

for $t = \mathsf{g}(\bar{t})$ and $s = \mathsf{f}(\bar{s})$

1.

or

2.

or

3.

Recursive Path Order (FOL)

Let $t >_{\mathsf{fo}} s$ if

for $t = \mathsf{g}(\bar{t})$ and $s = \mathsf{f}(\bar{s})$

1.

or

2. $\mathsf{g} \succ \mathsf{f}$,

or

3. $\mathsf{g} = \mathsf{f}$, $\bar{t} \gg_{\mathsf{fo}} \bar{s}$,

RPO's nature

Recursive Path Order (FOL)

Let $t >_{\mathsf{fo}} s$ if

for $t = \mathsf{g}(\bar{t})$ and $s = \mathsf{f}(\bar{s})$

RPO's duty
(for the subterm property)

1. $t' \geq_{\mathsf{fo}} s$ for some term $t' \in \bar{t}$

or

2. $\mathsf{g} \succ \mathsf{f}$,

or

3. $\mathsf{g} = \mathsf{f}$, $\bar{t} \gg_{\mathsf{fo}} \bar{s}$,

RPO's nature

Recursive Path Order (FOL)

Let $t >_{\mathsf{fo}} s$ if

for $t = \mathsf{g}(\bar{t})$ and $s = \mathsf{f}(\bar{s})$

RPO's duty
(for the subterm property)

1. $t' \geq_{\mathsf{fo}} s$ for some term $t' \in \bar{t}$

or

2. $\mathsf{g} \succ \mathsf{f}$,
and $t >_{\mathsf{fo}} s'$ for all $s' \in \bar{s}$

or

3. $\mathsf{g} = \mathsf{f}$, $\bar{t} \gg_{\mathsf{fo}} \bar{s}$,
and $t >_{\mathsf{fo}} s'$ for all $s' \in \bar{s}$

RPO's nature

Irreflexivity check

Applicative FOL

First-order signature
with only one function symbol **@**

@(@(map, *X*), @(@(cons,*Y*),*Z*))
= @(@(cons,@(*X*,*Y*)),@(@(map, *X*),*Z*))

Applicative FOL

First-order signature
with only one function symbol **@**

@(@(map, X), @(@(cons,Y),Z))
= @(@(cons,@(X,Y)),@(@(map, X),Z))

λ-free HOL

Higher-order terms
without λ-expressions

map X (cons Y Z)
= cons (X Y) (map X Z)

Applicative FOL

First-order signature
with only one function symbol **@**

@(@(map, *X*), @(@(cons,*Y*),*Z*))
= @(@(cons,@(*X*,*Y*)),@(@(map, *X*),*Z*))

RPO is weak because most heads are **@**

λ-free HOL

Higher-order terms
without λ-expressions

map *X* (cons *Y* *Z*)
= cons (*X* *Y*) (map *X* *Z*)

Applicative FOL

First-order signature
with only one function symbol **@**

@(@(map, *X*), @(@(cons,*Y*),*Z*))
= @(@(cons,@(*X*,*Y*)),@(@(map, *X*),*Z*))

RPO is weak because most heads are **@**

λ-free HOL

Higher-order terms
without λ-expressions

map *X* (cons *Y* *Z*)
= cons (*X* *Y*) (map *X* *Z*)

RPO using the “real” heads?

Recursive Path Order for λ-free HOL (Blanchette et al., 2017)

Let $t >_{\mathsf{ho}} s$ if

for $t = \xi\,\bar{t}$ and $s = \zeta\,\bar{s}$

1.

or

2.

or

3.

Recursive Path Order for λ-free HOL (Blanchette et al., 2017)

Let $t >_{\text{ho}} s$ if

for $t = \xi\,\bar{t}$ and $s = \zeta\,\bar{s}$

1.

or

2. $\xi \succ \zeta,$

or

3. $\xi = \zeta,\ \bar{t} >\!\!>_{\text{ho}} \bar{s},$

Recursive Path Order for λ-free HOL (Blanchette et al., 2017)

Let $t >_{\text{ho}} s$ if

for $t = \xi\,\bar{t}$ and $s = \zeta\,\bar{s}$

1. $t' \geq_{\text{ho}} s$ for some term $t' \in \bar{t}$

or

2. $\xi \succ \zeta$,

or

3. $\xi = \zeta$, $\bar{t} \gg_{\text{ho}} \bar{s}$,

Recursive Path Order for λ-free HOL (Blanchette et al., 2017)

Let $t >_{\mathrm{ho}} s$ if

for $t = \xi\, \bar{t}$ and $s = \zeta\, \bar{s}$

1. $t' \geq_{\mathrm{ho}} s$ for some term $t' \in \bar{t}$

or

2. $\xi \succ \zeta$,

and $t >_{\mathrm{ho}} s'$ for all $s' \in \bar{s}$

or

3. $\xi = \zeta$, $\bar{t} \gg_{\mathrm{ho}} \bar{s}$,

and $t >_{\mathrm{ho}} s'$ for all $s' \in \bar{s}$

Recursive Path Order for λ-free HOL (Blanchette et al., 2017)

Let $t >_{\text{ho}} s$ if

for $t = \xi\,\bar{t}$ and $s = \zeta\,\bar{s}$

1. $t' \geq_{\text{ho}} s$ for some term $t' \in \bar{t}$

or

2. $\xi \succ \zeta$, $\text{vars}(t) \supseteq \text{vars}(\zeta)$

and $t >_{\text{ho}} s'$ for all $s' \in \bar{s}$

or

3. $\xi = \zeta$, $\bar{t} \gg_{\text{ho}} \bar{s}$,

and $t >_{\text{ho}} s'$ for all $s' \in \bar{s}$

Only modification: condition for variable heads

Properties

- Coincides with first-order RPO on first-order terms
- Almost a ground-total simplification order,
  except for compatibility with arguments:

$$t >_{\mathsf{ho}} s \not\Rightarrow t\ u >_{\mathsf{ho}} s\ u$$

Properties

- Coincides with first-order RPO on first-order terms
- Almost a ground-total simplification order, except for compatibility with arguments:

$$t >_{\mathsf{ho}} s \not\Rightarrow t\ u >_{\mathsf{ho}} s\ u$$

Challenge: Find a ground-total simplification order for λ-free HOL that resembles RPO

Embedding

Embedding step $\rightarrow_{emb}$

Replace a subterm of the form **s t** by **s** or by **t**

For example:

$f\ (g\ a)\ (h\ b) \rightarrow_{emb} f\ g\ (h\ b)$

Embedding

Embedding step $\to_{emb}$

Replace a subterm of the form **s t** by **s** or by **t**

For example:

$f\ (g\ a)\ (h\ b) \to_{emb} f\ g\ (h\ b) \to_{emb} f\ g\ b$

Embedding

Embedding step $\rightarrow_{\text{emb}}$

Replace a subterm of the form **s t** by **s** or by **t**

For example:

$f\ (g\ a)\ (h\ b) \rightarrow_{\text{emb}} f\ g\ (h\ b) \rightarrow_{\text{emb}} f\ g\ b \rightarrow_{\text{emb}} f\ b$

Embedding

Embedding step $\rightarrow_{emb}$

Replace a subterm of the form **s t** by **s** or by **t**

For example:

$f\ (g\ a)\ (h\ b) \rightarrow_{emb} f\ g\ (h\ b) \rightarrow_{emb} f\ g\ b \rightarrow_{emb} f\ b \rightarrow_{emb}\ b$

Embedding

Embedding step $\rightarrow_{emb}$

Replace a subterm of the form **s t** by **s** or by **t**

For example:

$f\ (g\ a)\ (h\ b) \rightarrow_{emb} f\ g\ (h\ b) \rightarrow_{emb} f\ g\ b \rightarrow_{emb} f\ b \rightarrow_{emb} b$

Embedding relation $\triangleright_{emb}$

Transitive closure of $\rightarrow_{emb}$

For example:

$f\ (g\ a)\ (h\ b) \triangleright_{emb} f\ b$

Embedding

Embedding step $\rightarrow_{\text{emb}}$

Replace a subterm of the form **s t** by **s** or by **t**

For example:

$$\mathsf{f}\ (\mathsf{g}\ \mathsf{a})\ (\mathsf{h}\ \mathsf{b}) \rightarrow_{\text{emb}} \mathsf{f}\ \mathsf{g}\ (\mathsf{h}\ \mathsf{b}) \rightarrow_{\text{emb}} \mathsf{f}\ \mathsf{g}\ \mathsf{b} \rightarrow_{\text{emb}} \mathsf{f}\ \mathsf{b} \rightarrow_{\text{emb}} \mathsf{b}$$

Embedding relation $\rhd_{\text{emb}}$

Transitive closure of $\rightarrow_{\text{emb}}$

For example:

$$\mathsf{f}\ (\mathsf{g}\ \mathsf{a})\ (\mathsf{h}\ \mathsf{b}) \rhd_{\text{emb}} \mathsf{f}\ \mathsf{b}$$

Any simplification order has the embedding property:

$$t \rhd_{\text{emb}} s \Rightarrow t > s$$

Naive approach

Let $t >_{\text{epo}} s$ if

for $t = \xi\,\bar{t}$ and $s = \zeta\,\bar{s}$

1.

or

2.

or

3.

Naive approach

Let $t >_{\text{epo}} s$ if

for $t = \xi\,\bar{t}$ and $s = \zeta\,\bar{s}$

1.

or

2. $\xi \succ \zeta$

or

3. $\xi = \zeta,\ \bar{t} \gg_{\text{epo}} \bar{s},$

Naive approach

Let $t >_{\text{epo}} s$ if

for $t = \xi\, \bar{t}$ and $s = \zeta\, \bar{s}$

1. $t' \geq_{\text{epo}} s$ for some term t' with $t \to_{\text{emb}} t'$

or

2. $\xi \succ \zeta$

or

3. $\xi = \zeta,\ \bar{t} >\!\!>_{\text{epo}} \bar{s},$

Naive approach

Let $t >_{\text{epo}} s$ if

for $t = \xi\,\bar{t}$ and $s = \zeta\,\bar{s}$

1. $t' \geq_{\text{epo}} s$ for some term t' with $t \to_{\text{emb}} t'$

or

2. $\xi \succ \zeta$

and $t >_{\text{epo}} s'$ for all s' with $s \to_{\text{emb}} s'$

or

3. $\xi = \zeta$, $\bar{t} >\!\!>_{\text{epo}} \bar{s}$,

and $t >_{\text{epo}} s'$ for all s' with $s \to_{\text{emb}} s'$

Naive approach

Let $t >_{\text{epo}} s$ if

for $t = \xi\,\bar{t}$ and $s = \zeta\,\bar{s}$

1. $t' \geq_{\text{epo}} s$ for some term t' with $t \rightarrow_{\text{emb}} t'$

or

2. $\xi \succ \zeta$

and $t >_{\text{epo}} s'$ for all s' with $s \rightarrow_{\text{emb}} s'$

or

3. $\xi = \zeta$, $\bar{t} >\!\!>_{\text{epo}} \bar{s}$,

and $t >_{\text{epo}} s'$ for all s' with $s \rightarrow_{\text{emb}} s'$

Can be made to work

(with some modifications for applied variables)

Naive approach

But: Many terms to be checked

Let $t >_{\text{epo}} s$ if

for $t = \xi\,\bar{t}$ and $s = \zeta\,\bar{s}$

1. $t' \geq_{\text{epo}} s$ for some term t' with $t \rightarrow_{\text{emb}} t'$

or

2. $\xi \succ \zeta$

and $t >_{\text{epo}} s'$ for all s' with $s \rightarrow_{\text{emb}} s'$

or

3. $\xi = \zeta$, $\bar{t} >\!\!>_{\text{epo}} \bar{s}$,

and $t >_{\text{epo}} s'$ for all s' with $s \rightarrow_{\text{emb}} s'$

Can be made to work

(with some modifications for applied variables)

Chop

Definition:

$$\text{chop}(\zeta\ t_1\ \ldots\ t_n) = t_1\ \ldots\ t_n$$ (only defined if $n > 0$)

For example:

chop(f (g a) (h b)) = g a (h b)

Embedding Path Order

$t >_{\mathrm{ep}} s$ for $t = \xi\, \bar{t}_n$ and $s = \zeta\, \bar{s}_m$ if

1.

or

2.

or

3.

Embedding Path Order

$t >_{\text{ep}} s$ for $t = \xi\, \bar{t}_n$ and $s = \zeta\, \bar{s}_m$ if

1.

or

2. $\xi \succ \zeta,$

or

3. $\xi = \zeta,\ \bar{t}_n \gg_{\text{ep}} \bar{s}_m,$

Embedding Path Order

$t >_{\text{ep}} s$ for $t = \xi\, \bar{t}_n$ and $s = \zeta\, \bar{s}_m$ if

1. $n > 0$ and $\text{chop}(t) \geq_{\text{ep}} s$

or

2. $\xi \succ \zeta,$

or

3. $\xi = \zeta,\ \bar{t}_n \gg_{\text{ep}} \bar{s}_m,$

Embedding Path Order

$t >_{\mathsf{ep}} s$ for $t = \xi\, \bar{t}_n$ and $s = \zeta\, \bar{s}_m$ if

1. $n > 0$ and $\mathsf{chop}(t) \geq_{\mathsf{ep}} s$

or

2. $\xi \succ \zeta$,
   and either $m = 0$ or $t >_{\mathsf{ep}} \mathsf{chop}(s)$

or

3. $\xi = \zeta$, $\bar{t}_n \gg_{\mathsf{ep}} \bar{s}_m$,
   and either $m = 0$ or $t >_{\mathsf{ep}} \mathsf{chop}(s)$

Embedding Path Order

$t >_{\text{ep}} s$ for $t = \xi\, \bar{t}_n$ and $s = \zeta\, \bar{s}_m$ if

1. $n > 0$ and $\text{chop}(t) \geq_{\text{ep}} s$

or

2. $\xi, \zeta \in \Sigma,\ \ \xi \succ \zeta,$

   and either $m = 0$ or $t >_{\text{ep}} \text{chop}(s)$

or

3. $\xi, \zeta \in \Sigma,\ \ \xi = \zeta,\ \bar{t}_n \gg_{\text{ep}} \bar{s}_m,$

   and either $m = 0$ or $t >_{\text{ep}} \text{chop}(s)$

Embedding Path Order

$t >_{\text{ep}} s$ for $t = \xi\, \bar{t}_n$ and $s = \zeta\, \bar{s}_m$ if

1. $n > 0$ and $\text{chop}(t) \geq_{\text{ep}} s$

or

2. $\xi, \zeta \in \Sigma,\ \ \xi \succ \zeta,$
   and either $m = 0$ or $t >_{\text{ep}} \text{chop}(s)$

or

3. $\xi, \zeta \in \Sigma,\ \ \xi = \zeta,\ \bar{t}_n \gg_{\text{ep}} \bar{s}_m,$
   and either $m = 0$ or $t >_{\text{ep}} \text{chop}(s)$

or

4. $\xi, \zeta \in \mathscr{V},\ \xi = \zeta,\ \bar{t}_n \gg_{\text{ep}} \bar{s}_m,\ n > 0,$
   and either $m = 0$ or $\text{chop}(t) >_{\text{ep}} \text{chop}(s)$

Properties

- RPO-like, ground-total simplification order (verified in Isabelle)
- Worst runtime of size(s)*size(t)*depth(s)*depth(t) (implemented in Zipperposition)

Properties

- RPO-like, ground-total simplification order (verified in Isabelle)
- Worst runtime of size(s)*size(t)*depth(s)*depth(t) (implemented in Zipperposition)

Questions to you:

- Is it a known fact that any simplification order has the embedding property?
- Could this order be useful in other contexts outside of superposition?